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Counting Squares or Measuring Length? Mathematics Teachers' Hermeneutical Competence

Contare i Quadretti o Misurare la Lunghezza? La Competenza Ermeneutica degli Insegnanti di Matematica

¿Contar Cuadritos o Medir la Longitud? La Competencia Hermenéutica de los Docentes de Matemática

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Abstract. *This study explores the processes through which primary school teachers identify and interpret Grade 5 students' errors in relation to the INVALSI swordfish item. In the responses to this item, which was administered in an open-ended format to a sample of 100 students, a misconception confounding one-dimensional and two-dimensional quantities (length and area) appears recurrent. Representative student protocols of this error type were subsequently used as prompts during semi-structured interviews conducted individually with two primary school teachers; their responses were analyzed through the new theoretical lens of hermeneutical competence. The findings highlight different modes of expression of this competence and reveal how teachers' professional backgrounds can influence their processes of error identification and interpretation. Furthermore, the study shows how the guided analysis of student protocols and large-scale assessment data constitutes a promising formative tool to foster the development of teachers' hermeneutical competence.*

Keywords: hermeneutical competence, teacher education, misconceptions, large scale assessment.

Sunto. *Il presente studio esplora i processi attraverso cui gli insegnanti di scuola primaria identificano e interpretano gli errori degli studenti di Grado 5 in relazione all'item INVALSI del pescespada. Nelle risposte a tale quesito, somministrato in forma aperta a un campione di 100 studenti, appare ricorrente la misconcezione legata alla confusione tra grandezze unidimensionali e bidimensionali (lunghezza e area). I protocolli più rappresentativi di tale errore sono stati successivamente utilizzati come stimolo durante interviste semi-strutturate condotte individualmente con due insegnanti di scuola primaria; le trascrizioni sono state analizzate attraverso la nuova lente teorica della competenza ermeneutica. I risultati evidenziano differenti modalità* *di espressione di tale competenza e mettono in luce come il background professionale dei docenti possa influenzare i processi di identificazione e interpretazione dell'errore. Al contempo, lo studio mostra come l'analisi guidata dei protocolli e dei dati su larga scala costituisca un promettente strumento formativo per promuovere lo sviluppo della competenza ermeneutica degli insegnanti.*

Parole chiave: competenza ermeneutica, formazione insegnanti, misconcezioni, valutazione su larga scala.

Resumen. *El presente estudio explora los procesos mediante los cuales los docentes de educación primaria identifican e interpretan los errores de los estudiantes de Grado 5 en relación con el ítem INVALSI del pez espada. En las respuestas a dicha tarea, administrada de forma abierta a una muestra de 100 estudiantes, aparece como recurrente la idea errónea (misconcepción) vinculada a la confusión entre magnitudes unidimensionales y bidimensionales (longitud y área). Los protocolos más representativos de este tipo de error se utilizaron posteriormente como estímulo durante entrevistas semiestructuradas realizadas individualmente a dos docentes de educación primaria; las transcripciones se analizaron a través de la nueva lente teórica de la competencia hermenéutica. Los resultados evidencian diferentes modalidades de expresión de dicha competencia y muestran cómo la trayectoria profesional de los docentes puede influir en los procesos de identificación e interpretación del error. Al mismo tiempo, el estudio señala cómo el análisis guiado de los protocolos y de los datos a gran escala constituye una prometedora herramienta formativa para promover el desarrollo de la competencia hermenéutica de los docentes.*

Palabras clave: competencia hermenéutica, formación docente, concepciones erróneas, evaluación a gran escala.

1. Introduction

A crucial component of teachers' professional competence is based on the knowledge mobilized in making sense of students' mathematical reasoning, particularly when encountering non-standard responses or errors (Peng & Luo, 2009). This interpretative competence is essential not only for evaluating the correctness of a response but also for uncovering underlying cognitive processes and related misconceptions. Within this framework, large-scale assessments (LSA) can serve as powerful tools for research and teacher education (Bolondi & Ferretti, 2021), providing a rich context to investigate how teachers read and interpret students' mathematical productions (Cibien et al., 2025).

In continuity with this perspective, the present study is embedded within the broader national project *Mathematics standardized assessment as a tool for teachers' professional development* (*MaSt*–https://mast.unife.it/). The project's core objective is to develop theoretical and operative tools that adopt Italian LSA INVALSI results to support teachers' growth and promote a formative use of these materials at both institutional and classroom levels (Bolondi et al., 2025). Traditionally, the impact of LSA follows a top-down trajectory, where results influence public opinion and policymakers before finally reaching local teacher action (Looney, 2011). The innovative focus of the *MaSt* project is to reverse this dynamic through a bottom-up approach, providing tools that allow teachers to use LSA frameworks and results immediately to implement formative assessment and individualized instruction in their classroom. This approach aligns with the idea that assessment is a key component of a teacher's identity and practice (Ferretti & Santi, 2023); therefore, teachers must perceive a clear connection between assessment and the curriculum. As a preliminary step toward operationalizing this approach into a concrete professional development model for pre- and in-service teachers, the initial aim was to gather information on their beliefs and knowledge, in order to identify the educational needs to be addressed during the professional development program. To collect this information, an anonymous Google Forms questionnaire was administered to 473 mathematics teachers (both pre-service and in-service). The questionnaire investigated several aspects: how teachers perceive specific INVALSI items in terms of assessment, students' perceived difficulty, and consistency with national guidelines; whether and how they explain students' potential errors and their causes; and which factors, including items' formulation, might influence their difficulty. The survey results provide crucial insights, revealing that regardless of school level or experience, teachers often struggle to correctly interpret common students' errors (Cibien, 2025). Furthermore, an inconsistency exists between teachers' perception of an item's coherence with ministerial guidelines and their actual teaching practices. An example of this distance can be observed when items featuring problems perceived as non-standard, are viewed as detached from classroom routines due to their divergence from daily instructional practices; this highlights a discontinuity between the assessment framework and everyday teaching. These results served as the starting point for designing the professional development model (Bolondi et al., 2026).

Based on these premises, the study presented in this contribution was conducted with the objective of observing how mathematics primary teachers reflect on a specific INVALSI item that reveals a potential national-level macro-phenomenon (Ferretti & Bolondi, 2019). Specifically, the item under consideration (aimed at Grade 5 students) highlights a widespread confusion between the concept of measurement understood as length (the side of a square) and surface area (the area represented by the square itself). This phenomenon, already documented in the literature (Martini & Sbaragli, 2005), remains under-researched from the perspective of teacher analysis. The need to study teachers' awareness in this specific context arises because, when confronted with errors rooted in this length-area confusion, teachers may easily notice that an answer is wrong but struggle to explain the exact conceptual reason behind it. In this regard, Livy and colleagues found that many pre-service primary school teachers tend to rely on purely procedural descriptions (such as "length times width") without explicitly referring to the mathematical quantity involved or the underlying conceptual structures (Livy et al., 2012). Therefore, this paper focuses precisely on how teachers analyze this specific geometrical difficulty, using it as a key lens to explore their broader ability to interpret student errors.

Building on these insights, the present study focuses on understanding how teachers identify and interpret students' errors in a context where graphical representation, measurement, and proportionality interact, specifically within the context of an open-ended INVALSI item for primary school students. Indeed, this study aims to explore which potential errors primary school teachers identify when incorrect procedures or strategies are not pre-defined, in order to observe what types of errors teachers focus on and how they initiate a process of *coming to know* their underlying rationales. To analyze this specific interaction, teachers' responses are examined through the new lens of *hermeneutical competence* (Cibien, 2026). This construct allows for moving beyond the distinction between procedure and concept, offering a lens to observe different identificatory and interpretative abilities when teachers engage with authentic protocols that manifest the misconception between the side and the area of a square.

2. Theoretical framework

2.1. Misconceptions

The construction of mathematical knowledge is not a linear accumulation of information, but rather a complex process involving the continuous refinement of mental models (Tall & Vinner, 1981). According to Fischbein and colleagues (Fischbein et al., 1985), students enter the learning environment with alternative conceptions, internally coherent and often highly resilient ideas that may conflict with formal mathematical structures. A misconception emerges when a student's initial and stable mental image of a concept fails to accommodate new and more complex information, thereby generating internal cognitive conflict (Sbaragli, 2005). Rather than being understood merely as isolated errors, misconceptions can be viewed as relatively stable and locally coherent ways of thinking that reveal how students organize and extend their prior knowledge. From this perspective, they may become productive starting points for instructional intervention, error analysis, and conceptual restructuring when explicitly addressed in teaching (Ndemo & Ndemo, 2023; Skemp, 1976). In this perspective, many students' errors should not be interpreted as isolated failures, but rather as the expression of general intuitive rules that are spontaneously activated across different mathematical and scientific contexts (Stavy & Tirosh, 1996). Because these rules are initially functional and cognitively compelling, they may support the early construction of locally coherent interpretations which, if not progressively restructured, can stabilize into persistent misconceptions. This perspective is consistent with what Fischbein (1999) describes as the "coercive character" of intuitive models. Within this perspective, students' errors cannot be interpreted as isolated failures or simple inaccuracies, but rather as the consequence of intuitive models that have been constructed too early and subsequently stabilized through instruction (D'Amore et al., 2008). In many cases, these models are not formed independently by learners alone, but are also shaped, reinforced, or even inadvertently produced by the teacher's language, examples, and representational choices (Girit et al., 2023). When certain expressions are repeatedly used in a simplified or ambiguous way, students may attribute to them a meaning that is locally functional but conceptually inappropriate (Fandiño Pinilla & D'Amore, 2007). For instance, this linguistic and representational ambiguity is particularly frequent in the Italian educational context due to the pervasive use of squared notebooks. In daily classroom practice, teachers and students routinely refer to the individual grid units simply as "squares," blurredly using the same term to denote both the two-dimensional area of the grid square and the linear length of its side. Consequently, when asked to draw a figure with a side length of four units, it is common colloquial practice to say "a side of four squares" rather than specifying "four grid-square sides." This shortcut leads students to confuse linear and two-dimensional measures, thereby fostering a partial and misleading representation of the underlying concept. Once an intuitive model of this nature, such as the one exemplified above, has become stable, it tends to guide interpretation and problem solving with considerable persistence, generating recurrent errors that are resistant to correction. In this sense, misconceptions can be understood as the result of a premature stabilization of intuitive meaning, when a representation that is initially useful in a limited context becomes overgeneralized and continues to operate beyond its domain of validity (Fischbein et al., 1985).

In this regard, D'Amore (2022) highlights the crucial role of semiotic awareness in teaching, namely, the need for educators to recognize that students may interpret the same signifier, whether a graph, a symbol, or a technical term, in ways that differ radically from the teacher's intention. For example, a teacher may use ellipsis points to suggest the continuation of an infinite irrational sequence, while a student may instead perceive them as indicating a predictable or eventually terminating pattern. When instrumental reasoning and conceptual reasoning are aligned, learning tends to proceed smoothly (Sbaragli, 2005; Skemp, 1976). However, when the two levels diverge, the teacher's role is not simply to correct an "error," but rather to be aware of the situation and to help deconstruct the stable yet partial mental image the student has formed. By varying contexts and explicitly challenging intuitive rules, teachers can support the transformation of a misconception, or of a prematurely formed intuitive model, into a more sophisticated and enduring conceptual model.

2.2. Hermeneutical Competence

The concept of hermeneutical competence (Cibien, 2026) provides a framework for understanding teachers' processes of error identification and interpretation. This construct was recently introduced out of the need to describe the expertise required by mathematics teachers to identify a students' error and, simultaneously, interpret it in a manner coherent with the context—both global and local—in which the error occurred (e.g. type of task, the purpose for which it was designed, the mathematical topic considered, and the way it was assigned). The conceptualization of this construct is rooted in a framework for connecting theoretical perspectives, according to the approach proposed by Prediger and colleagues (Prediger et al., 2008), whereby connecting different theoretical lenses enriches the analysis of complex educational processes. In particular, a *coordinating* approach was adopted, thereby integrating different elements from existing, well-known theories in order to coordinate the two considered theoretical perspectives to enrich the analysis of these two complex and, at the same time, fundamental processes of professional teaching. Specifically, hermeneutical competence integrates components from the existing constructs of *diagnostic competence in error situations* (Heinrichs & Kaiser, 2018) and *interpretative knowledge* (Ribeiro et al., 2013).

Diagnostic competence in error situations (Heinrichs & Kaiser, 2018) refers to the specific component of teachers' diagnostic competence (Schrader, 2011) concerning the ability to "diagnose" student errors. In this context, the teacher's goal is not to implement diagnostic processes oriented toward subsequent corrective action, but rather to infer through informal or semi-formal methods (such as the analysis of written work or short questions), the reasoning that may have led to the error (Larrain & Kaiser, 2022).

Interpretative knowledge (Ribeiro et al., 2013), on the other hand, concerns the knowledge activated by mathematics teachers when making sense of students' reasoning; it extends beyond the analysis of explicit errors to include ambiguous, incomplete, or alternative solution approaches, regardless of the correctness of the final answer. This form of knowledge is observable when teachers analyze the variety of approaches taken by students, influencing both the interpretation of reasoning and the planning of subsequent instructional didactic actions (Ribeiro et al., 2016). A specific kind of such knowledge is the Semiotic Interpretative Knowledge (Asenova et al., 2023), which focuses on how teachers make sense of the semiotic complexities inherent in students' mathematical productions.

The construct of hermeneutical competence was therefore introduced to respond to the need for an integrative theoretical lens, aiming to deepen the processes of identifying and interpreting students' errors by mathematics teachers through a more hermeneutical approach. Indeed, the two existing constructs do not embrace this broader, hermeneutical view of both processes simultaneously. Interpretative knowledge is a specific lens for studying how teachers interpret students' reasoning, but it presents limitations for the objective proposed here: it does not focus specifically on the initial identification of the error and is considered within broader contexts where teachers deal with a wide range of responses, including correct ones, rather than concentrating solely on erroneous procedures. Furthermore, its nature as mere knowledge does not take into account crucial aspects, such as issues related to beliefs and experience; consequently, it was necessary to also integrate elements of diagnostic competence in error situations, which precisely goes beyond mere knowledge to embrace broader dimensions of competence and places greater emphasis on error identification. However, even within this theoretical framework, the hermeneutical process through which a true understanding of the error and its underlying rationale is reached by means of the diagnostic activities involved in these processes is not fully explored.

Therefore, by integrating these two aspects into a more hermeneutical perspective, the new construct of hermeneutical competence was introduced, still framed as a competence, following the conceptualization of Blömeke et al. (2020): hermeneutical competence is understood as a dynamic and situated continuum, within which teachers' professional dispositions—such as knowledge, beliefs, and motivation—are concretely activated in the specific processes of error identification and hermeneutical interpretation. It is defined as the practice-related competence required by a mathematics teacher to coherently identify and interpret mathematical errors based on the analysis of student-produced texts (Cibien, 2026). Here, "texts" are understood in a broad sense, encompassing productions that are not only written but also, for example, graphical or verbal. Furthermore, the concept of coherence entails that, when putting these processes into practice, teachers must integrate a global perspective (represented, in this context, by LSA and the foundations of mathematics education) with a local one (more closely linked to the classroom context and the student's reasoning models). Thus, the inherently hermeneutical nature of this competence emerges: the teacher does not merely detect the error but explains its profound underlying motivation in perfect harmony with both the reference context and the analyzed hermeneutical object, mobilizing knowledge, beliefs, and experiences with the ultimate goal of reaching a holistic understanding of the considered error. To clearly circumscribe these hermeneutical objects and avoid confusing the interpretative process with what is being interpreted (Figal, 2022), items from the INVALSI tests were selected as a starting point to activate these processes. Designed in line with the National Curriculum Guidelines (MIM, 2025), these items—both open-ended and multiple-choice—are structured so that the possible incorrect responses are not random, but systematically reflect specific conceptual nodes widely documented by research in mathematics education (Ferretti et al., 2023; Garuti et al., 2017). This rigorous methodological architecture allows not only for the mapping of macro-phenomena on a national scale (Bolondi & Ferretti, 2021), but also provides a reliable statistical measure of student competences.

Within this theoretical framework, a well-developed hermeneutical competence allows for an in-depth analysis of an item and the corresponding student production. An expert hermeneutical analysis indeed requires the teacher to be able to "read through" the structure of the item to anticipate the errors most likely to occur on a national scale, thereby reflecting a clear awareness of recurring difficulties within real classroom practices. Ultimately, it is precisely through this hermeneutical work—aimed at coming to know, understood as the holistic attainment of the comprehension of the error and its motivations—that the practice-related and intrinsically dynamic nature of this competence comes to light. In identifying and interpreting the reasoning that generate the error, teachers do not focus on isolated cases, but consider the multiplicity of situations that may arise, handling the complex interaction between the subjective relationship (of the students) with the hermeneutical objects and the (objective) mathematical content of the objects themselves.

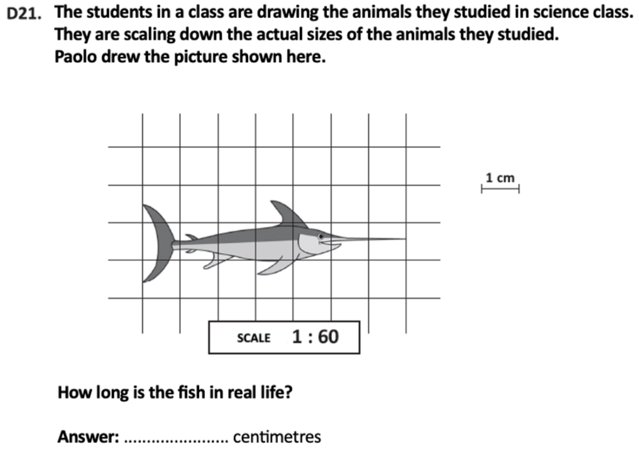
Starting from this perspective, the aim of the present investigation is to examine how this competence concretely manifests in the approaches of primary school teachers during the analysis of students' written protocols in relation to an open-ended INVALSI item.

3. The Study

The present study is exploratory in nature; it utilizes an open-ended item designed to shed light on how teachers' hermeneutical competence manifests, allowing them to express their interpretive reasoning and the ways in which they make sense of students' responses to open-ended tasks. Since it is a case-study, it adopts a qualitative design and a descriptive focus aimed at investigating few examples about how primary school teachers interpret students' written responses to a LSA item. Specifically, the research focuses on the INVALSI *swordfish item* (D21), administered to fifth-grade students in Italy in 2017. The item was purposefully selected because it aligns with a validated assessment model, providing a reliable and standardized scenario to draw out teachers' mathematical knowledge for teaching (Cibien et al., 2025). The item, shown in Figure 1, requires calculating the real length of a swordfish using a provided reduction scale (1:60) and its drawing on squared paper.

Figure 1

Item 21, INVALSI 2017 – Grade 5 (translation by the authors), available from Gestinv database www.gestinv.it



Although the requested calculation may at first glance appear to require only a simple multiplication, the mathematical activity involved is conceptually layered. Students must accurately read a graphical representation, identify the relevant unit of length on the grid, and interpret the multiplicative meaning of the scale factor before producing a numerical answer. Before applying the scale factor, students must first decide what exactly has to be measured in the drawing. The presence of squared paper introduces a significant interpretative uncertainty: the square may be considered as a discrete visual object, as a unit of area, or – through its side – as a linear unit. In this sense, the item requires a fundamental interpretative decision that precedes proportional reasoning itself. This type of difficulty has been widely documented in research on teaching and learning processes related to mathematical measurement. For instance, Kamii and Kysh (2006) showed that even when students work in apparently simple grid-based contexts, many do not spontaneously identify the square as a meaningful measurement unit. In their study, only a minority of younger students counted squares correctly, while many counted pegs, marks, or isolated visible elements instead, suggesting that graphical supports do not automatically generate a conceptual understanding of units. Their results also show that difficulties with area are closely linked to more general difficulties in understanding length units: many children counted visible markers on rulers rather than intervals, revealing uncertainty about what constitutes a measurable quantity. This point is particularly relevant for the swordfish item because students must interpret the grid not as an area structure, but as a support for linear measurement. The mathematical object to be considered is not the square itself, but the side of each square as a unit of length. It is explicitly noted that many students treat squares as rigid discrete objects rather than as units embedded in continuous measurement processes. Such findings help explain why a scale item may generate profound conceptual errors even when multiplication is available as an acquired procedural skill (Kamii & Kysh, 2006).

The choice of this specific item is also motivated by its performance at the national level, which highlights significant and widespread difficulties among students, although these topics are extensively covered in school curricula and align with the National Curriculum Guidelines (MIUR, 2012). Data from the Gestinv database (www.gestinv.it) show that the item had a low percentage of correct answers nationwide: only 29.3% of students answered correctly, 64.2% incorrectly, and 6.5% left it blank. The discrepancy between the apparent computational simplicity of the item and the high percentage of incorrect responses nationwide makes it an exceptionally rich item for investigating how teachers interpret and make sense of students' underlying mathematical difficulties.

The data collection was structured in two distinct, sequential steps. Initially, the selected protocols were collected by administering the item to a sample of 100 fifth-grade primary school students. This administration differed significantly from the standard INVALSI procedure: to make the cognitive processes visible, students were explicitly asked to explain their reasoning in writing (following the prompt "Write down your procedure" after giving the answer to the item). In addition, no time limit was imposed, and students were encouraged to use the entire area of the sheet to solve the item. The students' written protocols were collected and analysed, distinguishing between correct and incorrect protocols, which were further classified according to the strategies employed. The selected protocols are representative of the processes carried out by the participating students and can be framed within the theoretical framework of misconceptions, as will be analyzed in the next section. The chosen protocols were then implemented during the teacher interviews to investigate their reflections regarding them.

Accordingly, semi-structured individual interviews (Cohen et al., 2002) were conducted with two primary school teachers, during which they analysed the selected protocols "live". The two teachers were selected due to their contrasting educational backgrounds and varying years of teaching experience. Specifically, T1 is a primary unexperienced teacher who has just concluded his formal teacher education path, whereas T2 is an experienced primary in-service teacher with 23 years of service but no formal training in mathematics education. The distinct backgrounds of the interviewed teachers provided contrasting perspectives, thereby facilitating a comprehensive analysis of hermeneutical competence across different professional profiles. The interviews' dynamic setting allowed teachers to comment on the correctness of the responses, reconstruct the students' possible reasoning strategies, and identify the true nature of the observed errors. To ensure consistency while allowing for open-ended exploration, the semi-structured interviews were guided by the following questions:

Q1. *Which knowledge and competencies do you think are necessary to answer the item?*
Q2. *What do you think is the most frequent correct solving strategy?*
Q3. *What do you think is the most frequent error? Why?*
Q4. *Analyzing the students' protocols, we ask you to interpret their resolutions, telling us if they are expected or not.*
Q5. *How would you assess the students' correct protocols?*
Q6. *What do you think was the percentage of correct and incorrect responses to this item at the national level?*
Q7. *After seeing the national results, what are your impressions or reactions?*

Questions Q1 and Q2 serve as the baseline for the subsequent analysis of responses through the lens of hermeneutical competence. These initial queries investigate the teachers' immediate reflections regarding the selected item, specifically focusing on the knowledge and competencies students must deploy to solve it successfully, as well as the solving strategies teachers anticipate being the most frequent or standard. By doing so, it is possible to establish an understanding of the teachers' initial perception of the item's resolution. This baseline could then be contrasted with the insights emerging from subsequent answers, particularly those concerning student errors. This allows us to assess whether there is consistency between what teachers consider necessary for a correct resolution and what they identify within student errors. For instance, if a teacher posits that the required knowledge entails distinguishing between area and length, and subsequently interprets student protocols through this specific lens, a clear coherence emerges across his or her responses, thereby characterizing an active hermeneutical process. The question Q3 is explicitly designed to reveal how teachers identify and interpret potential student errors, making it a cornerstone for exploring the emergence of hermeneutical competence. Through this prompt, teachers' knowledge, perceptions, and beliefs about the most frequent errors (the process of error identification) and their underlying causes (the process of error interpretation) are brought to light. Similarly, Q4 delves deeper into the process by which teachers interpret the provided student protocols, moving beyond the errors they might have purely hypothesized in the previous phase. This query allows us to investigate the mechanism through which teachers attempt to comprehend actual student misconceptions. Concurrently, analysing the protocols may prompt a shift in the teachers' perspectives regarding potential student errors, leading to variations in how they consider and interpret them. Question Q5 was specifically included because a preliminary analysis of the students' protocols revealed a frequent and critical discrepancy between solving procedures and final resolution:

indeed, some students provided the correct final answer despite having followed an incorrect, illogical, or purely heuristic procedure. Conversely, other students applied a perfectly sound mathematical procedure but reported a wrong result. Asking teachers to evaluate these specific protocols allows researchers to observe what they prioritize in their assessment: the formal correctness of the final outcome or the mathematical validity of the underlying reasoning. While this assessment process is not inherently bound to hermeneutical competence *per se*, it was included to examine whether the coherence observed in the identification and interpretation of student errors carries over into this dimension, which constitutes a core component of teachers' diagnostic activities. With Q6, the objective is to further investigate the extent to which teachers can globally gauge student errors. By asking them to estimate the percentages of correct and incorrect responses at a national level, we can observe the "weight" they attribute to specific errors, namely how frequent and plausible they believe these errors to be on a macro-scale. Finally, Q7 explores how teachers reconcile their previous reflections with actual national data. Responses to this final question allow us to observe the degree of coherence with which teachers integrate all the information at their disposal, while simultaneously assessing whether confronting the national results induces a shift or modification in their initial perspective.

The selection of the interview format was guided by the goal of eliciting teachers' interpretative reasoning rather than simply their ability to judge answers as correct or incorrect. This approach aligns with previous research demonstrating that analysing authentic student work is particularly effective for making teachers' mathematical knowledge and competencies for teaching explicit. As Livy and colleagues (Livy et al., 2012) showed, teachers' interactions with student work can reveal whether they rely mainly on procedural recognition or whether they mobilize deeper conceptual explanations connected to student reasoning. By structuring the interviews in this way, teachers were able to articulate which aspects of the students' productions they considered central, providing direct access to the interpretative dimensions and conceptual structures underlying their interpretations, insights that would remain largely implicit in purely written assessments.

4. Data Analysis

4.1. Analysis of Students' Written Protocols

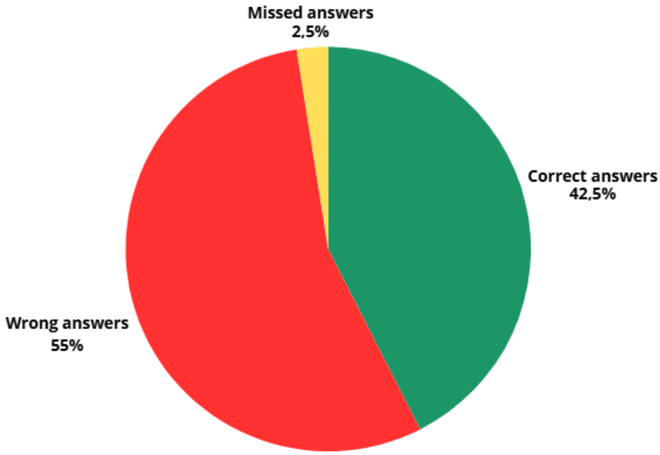
The analysis was conducted based on the swordfish item in Figure 1, utilizing an initial categorization of the responses into correct and incorrect cohorts. This was followed by a qualitative examination of individual protocols to identify specific solving processes and potential errors. Taking into account the broader framework of misconceptions, the specific objective was to identify instances where students confused "squares" with "squares' length" (sides of the squares).

The examination of specific protocols benefited from the integration of other theoretical lenses within mathematics education pertaining to both teaching and learning processes, such as the didactic contract (Brousseau, 1986), as it will be discussed in the next section.

Within the sample of 100 analyzed protocols, incorrect responses constituted the majority at 55%, while 42.5% were correct; a marginal 2.5% were classified as missing data or unjustified incorrect answers (Figure 2).

Figure 2

Percentage distribution of answers within the analyzed protocols



Regarding the incorrect responses, the study identified three primary macro-categories of error: the improper application of the conversion scale; the discrete counting of grid squares occupied by the figure in both length and height; and the calculation of the area of the rectangle bounding the subject. These findings are particularly significant as they highlight potential cognitive obstacles in isolating the one-dimensional attribute of length within two-dimensional spatial representations (Figures 3 and 4). Consequently, this article focuses specifically on the protocols manifesting such difficulties in the conceptualization of length, moving beyond mechanical errors related to numerical scales to explore the deeper semiotic and geometric challenges students encounter in grid-based environments.

Figure 3

Protocol 1 (Translation by the authors: "1+1+1+1+1+1+1+1+1+1+1+1+1 +1+1+1+1+1+1+1+1=21 length of the swordfish")

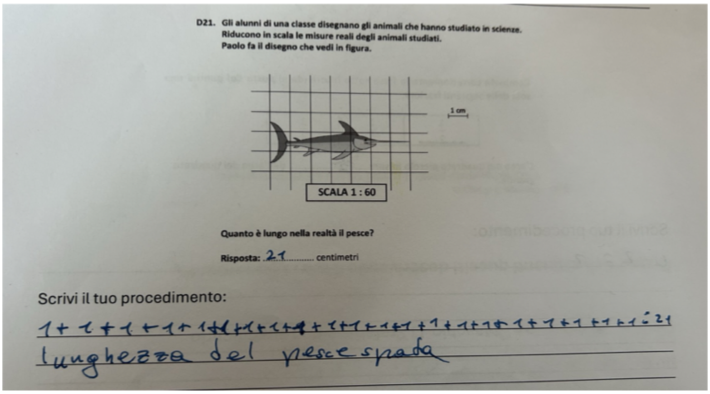
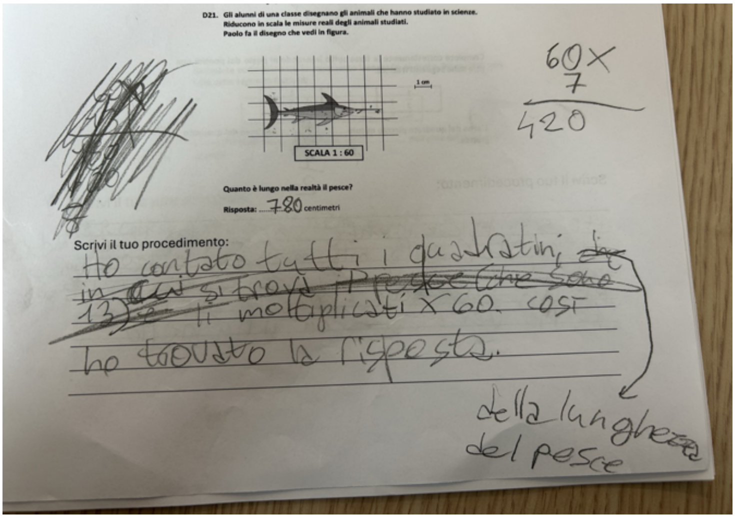


Figure 4

Protocol 2 (Translation by the authors: "I counted all the little squares of the length of the fish (see arrow) ~~that in which the fish is located (which are 13)~~ and I multiplied them × 60. So, I found the answer.")



Investigation into the correct protocols, by contrast, revealed diverse trajectories of reasoning. These included transitional strategies where the conversion scale was formally integrated into a mathematical expression but functioned merely as a symbolic division without impacting the actual numerical result (Figure 5). However, most correct responses demonstrated a clear developmental process based on identifying a length of 7 cm in the drawing and subsequently converting it into the real-world dimension of 420 centimetres or 4.20 meters (Figure 6).

Figure 5

Protocol 3 (Translation by the authors: "7(1:60)×60=420 cm equal to 4.20")

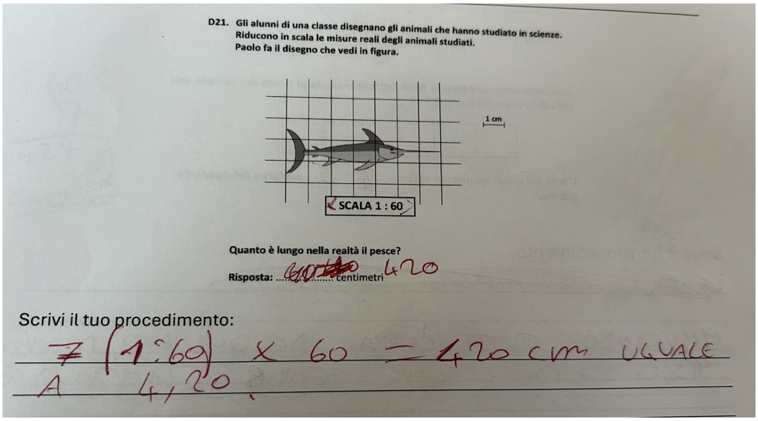
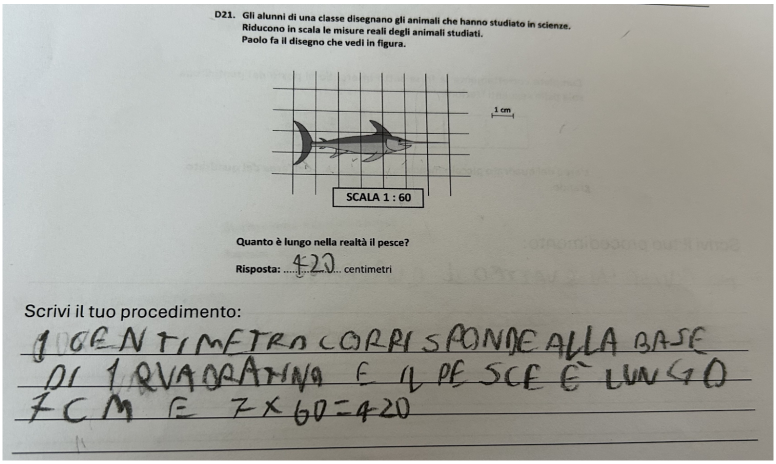


Figure 6

Protocol 4 (Translation by the authors: "1 centimeter corresponds to the base of one square and the fish is 7cm long and 7×60=420")



The qualitative analysis of students' protocols highlights the challenges outlined above.

Protocol 1 (Figure 3) and Protocol 2 (Figure 4) exemplify a significant potential cognitive obstacle, namely, the confusion between area and length. In the first case, the student calculates the area of the rectangular region enclosing the figure (7 cm by 3 cm), thereby substituting a surface measurement for a linear one. The reliance on repeated addition rather than multiplication may also suggest a difficulty in the transition from additive to multiplicative reasoning.

Protocol 2, despite correctly applying the conversion scale, reveals a similar underlying strategy based on counting and summing all the grid squares touched by the figure. This approach is clearly evidenced by the student's pencil marks, which indicate that each individual square was physically marked in order to keep track of the total count. This kind of behaviour is often reinforced by a lack of semiotic awareness in classroom practice. The ambiguous use of the term "square," instead of "side of the square", may create a linguistic overlap that contributes to the consolidation of distorted mental models. In these cases, students abandon the use of the scale and proportional reasoning, relying instead on the discrete presence of grid units as an inappropriate tool for determining a linear measure. From the theoretical perspective of misconceptions and intuitive models (Fischbein, 1999; Sbaragli, 2005), this interpretative focus becomes particularly relevant, as it shows how the repeated use of the same term in classroom discourse, often employed as if it were synonymous with a different concept, may gradually lead students to construct an erroneous representation. What appears linguistically economical from the teacher's perspective may therefore become cognitively misleading for the learner, fostering the premature stabilization of an intuitive but mathematically inappropriate model.

Furthermore, the analysis of the correct protocols reveals a spectrum of reasoning ranging from instrumental understanding to relational understanding (Skemp, 1976). Protocol 3 (Figure 5), for example, presents a correct numerical result achieved through a formally flawed procedure, in which the conversion scale appears primarily as a symbolic artifact. This behaviour can be framed, for instance, through the lens of the didactic contract (Brousseau, 1986), as the student appears to respond to what is perceived as the teacher's expectation by "reconstructing" the external form of a formally acceptable procedure, a practice typically emphasized in Grade 5. By explicitly including the scale, the student signals that the calculation is justified, even though the underlying proportional reasoning does not appear to be fully internalized.

By contrast, Protocol 4 (Figure 6) demonstrates a solid understanding of the concept of length and a sophisticated use of the grid. In this case, the student explicitly distinguishes the side of the square from the square itself, indicating that the early-stage misconception between linear and surface dimensions has been successfully overcome. Interestingly, the use of the term "base" in this protocol could suggest the persistence of residual positional misconceptions. However, interview data confirm that the student's functional understanding is secure. When asked for clarification, the student pointed to the specific segment on which the figure rests, suggesting the possible persistence of a positional misconception, namely, a failure to recognize the equivalence of any side of the square as a valid linear unit.

Ultimately, the progression from Protocol 1 to Protocol 4 illustrates the shift from a fragile, intuition-driven representation to a stable mathematical construct. These findings underscore the need for teachers to move beyond purely mechanical correction. By recognizing the strength of intuitive rules and demonstrating awareness of the factors that may influence the procedures evidenced in students' protocols, including semiotic factors related to the relationship between mathematical signs, language, and meaning (Duval, 2006), teachers can facilitate the deconstruction of any partial mental representations and support the development of a coherent and scientifically grounded mathematical perspective. Consequently, the theoretical conceptualization of the obtained protocols established the groundwork for the interviews, functioning as an effective tool for investigating the teachers' hermeneutical competence.

4.2. Analysis of Teachers' Interviews

The interviews conducted with the two teachers revealed points of convergence in the interpretation of the analysed errors, as well as significant divergences regarding the two processes under consideration (identification and interpretation). In this section are presented the emerged reflections, identifying the interviewed teachers as T1 and T2 as mentioned above. It is specified that, although the interviews were administered individually, for reasons of expository fluidity, the responses will be presented and compared by thematic nuclei (question by question) rather than by individual subjects. All the response excerpts reported were translated from the original language by the authors.

Regarding the first question (Q1), both teachers explicitly referred to both students' knowledge and competencies. Specifically, T1 outlined a framework of knowledge linked to the concept of scale and the ratios between quantities, while T2 focused exclusively on the aspect of scale. It is noteworthy that both interviewees emphasized that this conceptual node is primarily introduced in the primary school curriculum within the subject of geography, only later to be formalized and structured in a mathematical context. Concerning competencies, T2 explicitly recalls the "spirit of observation": students should be able to perceive that, although the anatomical components of the animal do not mirror known geometric figures, they must be considered in their entirety, even where the figure does not coincide exactly with a full square (referring to the "deception of the round tail", where the tips might be neglected because they occupy only a fraction of the grid unit). T1, instead, introduces the concept of the "three-dimensionality of the image", hypothesizing that it might lead students to erroneously consider the space occupied by the square rather than the measurement of the sides necessary for linear calculation. In this response, one can detect a reference, albeit implicit, to the misconception found in the students' protocols (which had not yet been shown to the teacher at this stage), despite the use of technically improper terminology: the figure indeed represents a three-dimensional object projected in a two-dimensional form. As will be observed subsequently, T1 will interpret numerous protocols through this constant reference to the aforementioned misconception, unlike T2. This preliminary discrepancy shows that the two teachers' hermeneutical competence manifests through different modalities.

Continuing the analysis, in responding to Q2 both teachers indicated the only expected correct strategy to be the product of the length detected on the grid (7 units) and the scale factor (60), without envisioning alternative solution approaches. Regarding Q3, despite the request to indicate a single type of error, both teachers proposed multiple options. T1 lists several possibilities: an "incorrect reading of the image" (omission of parts of the fish or, conversely, counting all the squares affected by the extension of the figure – a term that here replaces the previous "three-dimensionality") or confusion regarding the 1:60 ratio. In the latter case, T1 makes an explicit reference to the didactic contract (Brousseau, 1986): the presence of the division symbol in the scale notation could induce students to divide the correct number of measured units (7) by the scale value (60). Nevertheless, T1 considers an error in counting the units more probable, confirming a focus on the square rather than the side. This further statement underscores the consistency with which T1 remains anchored to his initial framing of the item. Moreover, another noteworthy element emerges from his interpretations that may underlie his hermeneutical competence: he appears to possess knowledge of certain well-established constructs in the literature. This factor undoubtedly influences his hermeneutical competence, as it allows him to interpret student errors not solely based on personal inclinations or beliefs, but by referring to robust constructs. T2 does not seem to display the same awareness; however, the interpretations proposed by T1 regarding an incorrect square count or a lack of understanding of the scale were in line with what was expressed by T2 in response to the same question: in fact, she stated "Either they didn't consider the front part of the animal, the elongated nose, or someone was misled by the tail: they didn't consider the fins. Then, probably someone didn't fully understand the relationship in the scale of reduction. This could also be a plausible error for children.". Therefore, both teachers appear to adopt a hermeneutical approach to the situation: even attempting to hypothesize multiple incorrect response strategies, they seek to provide a rationale for such behaviors by referring to various aspects, including those related to the choice of the proposed figure and the mathematical content of the item.

The analysis of the students' protocols highlights a further heterogeneity in the reflections of the two teachers. T1, consistently with his preliminary analysis, explicitly interprets the error as a misconception between "length and area" (i.e., between the side and the surface of the square), arguing that the student reasons in terms of the "positioning" of the fish on the grid rather than linear length. A particularly significant observation made by T1 concerns the students' tendency to revert to a "standard" situation, circumscribing the image within a known geometric figure (in this case, a 7×3 rectangle); according to the teacher, such a strategy stems from the inability to manage the analysis of a non-standardized real figure. Similarly, T2 notes that the student considers portions external to the figure yet interprets this reasoning exclusively as a gap in the concept of length. Unlike T1, T2 also focuses attention on the "limited knowledge of operations", referring to the use of repeated addition instead of multiplication. This aspect, although indicative of a hermeneutical approach aimed at holistically understanding the causes of error, highlights a shift from what was initially discussed by T2 in response to question Q1, a stage in which no considerations regarding operations had emerged. This discrepancy suggests that T1's point of view reshapes itself dynamically upon contact with the protocols, embracing possibilities not anticipated a priori. While on the one hand this manifests the situated and evolving nature of hermeneutical competence, on the other hand it highlights a discontinuity between the different phases of the interview. Since the focus on operations was not anticipated in the a priori analysis and is not further addressed in the subsequent stages, this reflection seems to take the form of an extemporaneous intuition, stimulated by the evidence of the specific protocol rather than by a predefined interpretative framework.

In relation to Protocol 3, T2 expresses greater perplexity than T1, stating difficulty in decoding a reasoning expressed in this incorrect mixed mode (verbal and symbolic): "The initial part of the protocol leaves me a bit perplexed, that 7 (1:60), even though the final result is correct.". Conversely, T1 interprets this formulation as the student's attempt to use "*matematichese*" (an informal term used to refer to formal symbolic mathematical language, intended as "Math-talk"), recalling formalisms and notations typical of school practice: "It's a bit as if they had transposed their logical process written in Math-talk. Yes, that is what I would say. And they didn't have anything else, meaning they didn't want to write it out in words, they wanted to write it like this.". In this case as well, the different ways in which the two teachers approach the protocol highlight varying degrees of hermeneutical depth. Indeed, the essence of this competence lies in the interpretative effort oriented toward *coming to know*—that is, toward unveiling the underlying rationale behind the student's thinking, regardless of the error situation presented. While T1 nevertheless proposes an interpretative hypothesis, T2's reaction highlights an area of potential development for the construct: an evolution aimed at consolidating the ability to cope with unexpected interpretative contexts, supporting the teacher in the attempt to give meaning to responses that, at first glance, are more difficult to "decipher". In any case, both teachers evaluate the resolution as correct (T1) or partially correct (T2).

Finally, the analysis of the last protocol – proposed to verify whether the student's explicit reference to the side of the square (defined as the "base") would stimulate a discussion by the teachers regarding the misconception under study – produced diametrically opposed reflections. T1 links back to the previously hypothesized confusion between square and side, focusing on the student's lexical choice ("The base tells me! Interesting: he counts under the fish and not above the fish. It's right, but it would have been better to say 'side'."). T2, by contrast, does not delve into these aspects, limiting herself to validating the student's statement ("Very true when he says the base! Totally correct."). This passage highlights even more clearly the coherence maintained by T1 throughout the entire interview, within the interconnected processes of identifying and interpreting errors. Conversely, T2's approach appears more focused on individual responses, seeming to lack a unified conceptual lens that would orient it toward the expected interpretative line.

In the final instance, regarding the national percentages, both teachers underestimated the difficulty of the item (in terms of the percentages of correct/incorrect answers), assuming a correct response rate slightly below 50%. Faced with the actual data, the reactions were divergent: T1 justified the observed complexity by noting that, in the national sample (where the item was open-ended but without a request for justification), all responses resulting from mere calculation errors are classified as incorrect, even if the underlying procedure might have been correct. However, T1 expressed marginal interest in the statistical data itself, prioritizing its professional impact: "The only interest is that I discover many got it wrong and therefore it can be worked on" (translation by the authors), highlighting the importance of LSA results as tools for conscious pedagogical practice. T2 did not provide analytical comments on the quantitative data, recognizing the students' protocols as a more effective tool for stimulating the professional reflections that emerged during the interview.

5. Discussion

The interviews proved to be an extremely effective tool for delineating various aspects of the hermeneutical competence of the teachers involved, as previously observed in other studies conducted within the same project (Cibien, 2026). Beyond confirming prior findings, the present study allows for a more fine-grained characterization of how such competence is enacted in situ, particularly in relation to the dynamic interplay between error identification, interpretation, and coherence across interpretative phases. Consistent with the findings discussed above, it can be stated that the two teachers manifest different facets of this competence. The primary distinction lies in the fact that T1, throughout the entire process of reflecting on items, protocols, and national data, exhibits greater consistency with his initial hypotheses, both in the identification and interpretative phases of the errors. In particular, his analyses appear strictly aligned with the examination of the question conducted through the theoretical lens of the "side-versus-area of the square" misconception, which served as the general framework. This coherence suggests not only the presence of relevant mathematical and didactical knowledge, but also the ability to mobilize it in a stable and context-sensitive way, a key feature of professional competence as described by Blömeke et al. (2020). Therefore, it can be argued that the coherence displayed by T1 throughout every phase of the interview, together with the hermeneutical approach implemented in the analysis of the various student productions, appears to reflect a solid and well-structured manifestation of his hermeneutical competence.

Conversely, teacher T2 appears less anchored to this prevailing interpretation, manifesting a plurality of perspectives during the identification and analysis of the protocols that do not always appear interconnected. Unlike T1, T2 does not proceed from a core idea to develop it consistently across the different stages of the interview. Notably, even when the reference to the issue of the side of the square becomes explicit (as in Protocol 4), the teacher does not produce structured reflections on the matter, limiting herself to confirming the correctness of the student's reasoning. This behavior suggests a more episodic activation of interpretative resources, in which relevant insights emerge locally but are not integrated into a coherent explanatory framework.

From this perspective, the comparison between T1 and T2 highlights a crucial dimension of hermeneutical competence: not merely the ability to produce plausible interpretations, but the capacity to maintain interpretative coherence across different moments of analysis. Such coherence appears to function as an organizing principle that allows teachers to connect individual observations to broader conceptual structures, thereby transforming isolated insights into professionally meaningful interpretations.

It is interesting to observe how, especially in the case of T2, elements not previously articulated emerge during the interpretation of the protocols yet are deemed relevant for the required solution. For example, the teacher highlights a critical issue in Protocol 1 related to the student's (lack of) awareness regarding the operation employed. However, this aspect had not been identified as relevant in response to question Q1. This discrepancy suggests that certain components of teachers' competence may remain implicit until activated by concrete artefacts such as students' productions. At the same time, it reveals a potential fragility in the structuring of such competence, which may not yet be fully integrated into a stable interpretative schema.

Furthermore, the analysis reveals how teachers' interpretations may sometimes be influenced by general pedagogical beliefs that are not entirely aligned with the specific mathematical situation. In the case of T2, the emphasis on the use of repeated addition instead of multiplication is interpreted as a lack of operational competence, whereas, in the context of the protocol, it may instead represent a faithful trace of the student's counting strategy. This misalignment underscores the importance of situating interpretations within the specific semiotic and conceptual features of the task, rather than relying on generalized expectations about students' performance. This element is of particular interest as it reflects the complex articulation of the components structuring the construct of competence adopted in the present work. Indeed, hermeneutical competence, as such, is shaped by the interaction among teachers' knowledge, beliefs, and dispositions; in the specific scenario described, a clear prevalence of the teacher's beliefs regarding the reasons underlying the student's solution choice can be observed. This seems to hinder the consideration of alternative interpretations and inhibit the activation of other forms of knowledge, such as Semiotic Interpretative Knowledge (Asenova et al., 2023), which would have allowed for a more in-depth hermeneutical analysis. The centrality of the affective and evaluative dimension, linked to beliefs and presumably to experiential factors, emerges transversally across multiple points in the interviews.

In several passages, in fact, a certain perplexity emerges regarding the protocols presented (particularly with respect to Protocol 3), perceived as unexpected lines of reasoning or difficult to contextualize within daily classroom practice. It is hypothesized that such "amazement" may derive from a strong anchoring to preconceived solution models or to limited exposure to non-standard student responses, if not indeed from a reliance on their own professional experiences. In the specific case of T2, who had expressed significant perplexity regarding that protocol, it can be assumed that her extensive teaching experience may have influenced her perspective, as it might represent an error, she never had the opportunity to encounter directly during her classroom practice. Although not formally explicit by the teacher, such an influence of prior experience constitutes a plausible hypothesis that is widely documented in the literature, which highlights how field practice shapes and sometimes constraints teachers' expectations when faced with unusual strategies (Superfine, 2009). Consequently, this finding reinforces the importance of incorporating authentic and varied student productions into teacher education programs, as they can challenge implicit assumptions and broaden teachers' interpretative repertoires.

More generally, it can be observed how the different backgrounds of the interviewed teachers may have influenced their respective reflections: on the one hand, T1, having completed their studies more recently, presumably mobilized knowledge and skills acquired during their training path; on the other hand, T2 appeared to be more closely anchored to beliefs and knowledge stemming from her own teaching experience. This highlights how all these components contribute fundamentally to shaping mathematics teachers' hermeneutical competence, confirming its nature as a dynamic competence that is strictly practice-related.

Furthermore, the use of protocols and resources derived from LSAs in training contexts thus appears particularly valuable. In this study, such tools proved effective not only for observing the manifestation of hermeneutical competence but also for fostering its potential development. By engaging with concrete student responses, teachers are prompted to make explicit their interpretative criteria, confront alternative reasoning paths, and refine their understanding of underlying misconceptions.

Finally, it is noteworthy that reflections on national percentages were perceived as less impactful compared to the analysis of individual protocols. While LSA data provide a valuable overview of students' performance, they appear less effective in supporting the interpretation of cognitive processes. This suggests that, within professional development models, quantitative data should be complemented with qualitative materials that allow teachers to access the reasoning behind errors. In this sense, the integration of LSA results with authentic student work emerges as a key condition for promoting a deeper and more functional development of hermeneutical competence.

6. Conclusion

The investigation conducted in this contribution suggests that hermeneutical competence represents a central resource for teaching professionalism, particularly in relation to the interpretation of students' mathematical thinking. More specifically, the study highlights that such competence cannot be reduced to the correct identification of errors, but involves the ability to construct coherent and context-sensitive interpretations that connect students' productions to underlying conceptual structures.

Within this perspective, the findings reinforce the importance of designing professional development pathways that explicitly target the hermeneutical dimension of teaching. The initial steps taken within the professional development model of which this study is a part (Bolondi et al., 2026) move in this direction, aiming to structure activities that encourage the emergence and refinement of hermeneutical competence through the combined use of student protocols and LSA data. The results of this study suggest that particular emphasis should be placed on supporting teachers in developing coherence across interpretative phases, as well as in integrating different sources of information into a unified analytical framework.

Although these are preliminary data, the results reveal different ways in which hermeneutical competence manifested, emphasizing the need to consciously support the processes through which teachers identify and interpret students' errors. In particular, the comparison between the two case studies shows that even when teachers are able to produce locally plausible interpretations, the absence of a stable interpretative structure may limit the depth and effectiveness of their analyses. This finding points to the necessity of fostering not only knowledge acquisition, but also the organization and mobilization of such knowledge in practice.

A further aspect of particular interest concerns the emergence of new training needs. The interviews highlight teachers' interest in exploring alternative task configurations and instructional scenarios starting from authentic student responses. For instance, the proposal advanced by T1 to modify the swordfish item (e.g., changing the orientation of the figure or using more standardized representations) reflects a productive shift from analysis to instructional design. This suggests that the development of hermeneutical competence may also support teachers in anticipating students' difficulties and in designing tasks that make specific conceptual aspects more visible.

From a broader perspective, the study contributes to the ongoing discussion on the formative use of LSA. While LSA data are often used primarily for evaluative or accountability purposes, the findings suggest that their integration with qualitative analyses of student work can significantly enhance their educational value. In particular, such integration allows for bridging the gap between macro-level data and classroom-level practices, supporting a more reflective and informed use of assessment.

Finally, the study is subject to some limitations. The small sample size, limited to two case studies, does not allow for generalization of the findings. However, the depth of the qualitative analysis provides valuable insights into the processes underlying teachers' interpretations. Future research will aim to expand the sample and to further investigate how hermeneutical competence develops over time, particularly within structured professional development programs. Additionally, it would be valuable to explore how this competence interacts with other dimensions of teachers' professional knowledge, such as pedagogical content knowledge and beliefs about teaching and learning.

In conclusion, this study highlights the relevance of focusing on teachers' interpretative practices as a key lever for improving mathematics education. Supporting the development of hermeneutical competence appears to be a promising direction for fostering more meaningful engagement with students' thinking, ultimately contributing to more effective and responsive teaching practices.

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